International Journal of Applied Mathematics
& Statistical Sciences (IJAMSS)
ISSN(P): 2319-3972; ISSN(E): 2319-3980
Vol. 6, Issue 6, Oct – Nov 2017; 17-22

International Academy of Science, Engineering and Technology
Connecting Researchers; Nurturing Innovations

FINSLER SPACE WITH (α, β) - METRIC

GAYATHRI K
Department of Mathematics, REVA University, Bengaluru, Karnataka, India

ABSTRACT

In Finsler space we see special (α, β) –metrics, such as Randers metric, Kropina metric and Matsumoto metric. etc. Locally dully at Finsler metrics arise from Information Geometry. Such metrics have special geometric properties and will play an important role in Finsler geometry. In this paper, we are going to study a class of locally dually at Finsler metrics which are defined as the sum of a Riemannian metric and 1-form. In this paper, we study the special (α, β) - metric L satisfies $L^2(\alpha, \beta) = 2\alpha^2 + \alpha\beta + 2\beta^2$, where

p

c^i are constants, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a di erential 1 form.

KEYWORDS: Finsler Space, (α, β) -Metric, Special (α, β) -Metric, Locally Dually Flat Metric, Flag Curvature

AMS Subject Classification (2010): 53C60, 53B40.

1. INTRODUCTION

Finsler spaces are the most natural generalization of Riemannian space, Finsler space is considered as space in which the line element is a function of positive homogeneity, which was initiated by P. Finsler.

The concept of (α, β) -metric was introduced in 1972 by M. Matsumoto and studied by M. Hachiguchi (1975), Y. Ichijyo (1975), S. Kikuchi (1979), C. Shibata (1984). The examples of the (α, β) -metric, are Randers metric, Kropina metric and Matsumoto metric. Z. Shen extended the notion of dually atness [9] to Finsler metrics.

2. PRELIMINARIES

A Finsler metric on a manifold M is a C^1 function F: T M! [0; 1) satisfies the following properties:

Regularity: L is a C1 on TM;

Positively homogeneity: L (x, y) = L (x; y); for λ> 0;

Strong convexity: The fundamental tensor $g_{ij}(x; y)$ is positive for all (x; y).

The geodesics of a Finsler space $F^n = (M^n, L)$ are given by the differential equations:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

Where $2G^i(x,y) = \gamma_j^i{}_k(x,y)y^j y^k$ and $\gamma_j^i{}_k(x,y)$ are Christoffel symbols constructed from $g_{ij}(x,y)$ with respect to x^i.

A Finsler space F^n is said to be of Douglas type if

$$D^{ij} \equiv G^i(x,y)y^j - G^j(x,y)y^i \tag{2.1}$$

Are homogeneous polynomials in (y^i) of degree three.

Let $hp(r)$ denote the homogeneous polynomial in y^i of degree r.

We use the following definition in future.

Definition 2.1

The Finsler space F^n is of Douglas type if and only if the Douglas tensor

$$D_{i\ jk}^{h} = C_{i\ jk}^{h} - \frac{1}{n+1}(G_{ijk}y^h + G_{ij}\delta_k^h + G_{jk}\delta_i^h + G_{ki}\delta_j^h)$$

Vanishes identically, where $G_{i\ jk}^{h} = \dot{\partial}_k G_{i\ j}^{h}$ is $h\upsilon$ − curvaturetensor of the Berwald connection.

The covariant differentiation with respect to the Levi-Cavita connection $\{{}_j{}^i{}_k\}(x)$ of R^n is denoted by(|). We use the symbols as follows:

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \qquad s_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i})'$$

$$s_j = b_r s^{rj}, \qquad s_j^i = a^{ir} s_{rj}¶$$

The functions $G^i(x,y)$ of F^n with (α,β) − metric are written in the form

$$2G^i = \left\{{}_0{}^i{}_0\right\} + 2B^i,$$

$$B^i = \frac{\alpha L_\beta}{L_\alpha} s_0^i + C^* \left[\frac{\beta L_\beta}{\alpha L} y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}\left(\frac{1}{\alpha} y^i - \frac{\alpha}{\beta} b^i\right)\right], \tag{2.2}$$

Where $L_\alpha = \partial L/\partial\alpha, L_\beta = \partial L/\partial\beta, L_{\alpha\alpha} = \partial L/\partial\alpha\partial\alpha,$ the subscript 0 means contraction by y^i and we put

$$C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})},$$

Where $\gamma^2 = b^2\alpha^2 - \beta^2, b^i = a^{ij}b_j \ and\ b^2 = a^{ij}b_i b_j.$

Since $\left\{{}_0{}^i{}_0\right\}(x)$ are $hp(2), F^n$ with (α,β) − metric is a Douglas space if and only if $B^{ij} \equiv B^i y^j - B^j y^i$ are $hp(3)$. From (1.2.1) and (1.2.2), we have

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s_0^i y^j - s_0^j y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^*(b^i y^j - b^j y^i) \tag{2.3}$$

We use the following lemma.

Lemma 2.1

If $\alpha^2 \equiv 0(\mathrm{mod}\beta)$, that is, $a_{ij}(x)y^i y^j$ contains $b_i y^i$ as a factor, then the dimension n is equal to 2 and b^2 vanishes. In this case, we have 1-form $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.

2.2. DOUGLAS SPACE WITH SPECIAL METRIC $L = \alpha - \frac{\beta^2}{\alpha}$

In this section, we find the condition for a Finsler space F^n With a special $(\alpha, \beta) -$ metric

$$L = \alpha - \frac{\beta^2}{\alpha} \quad \textbf{(3.1)}$$

To be a Douglas type. The derivatives of (3.1) are given by

$$L_\alpha = 1 + \frac{\beta^2}{\alpha^2}, \quad L_\beta = -\frac{2\beta}{\alpha}, \quad L_{\alpha\alpha} = -\frac{2\beta^2}{\alpha^3} \quad \textbf{(3.2)}$$

Substituting (3.2) in (2.3), we get

$$\{\alpha^2(1-2b^2)+3\beta^2\}\{(\alpha^2+\beta^2)B^{ij} - \alpha^2(\alpha-2\beta)(s_0^i y^j - s_0^j y^i)\} + \alpha^2\{r_{00}(\alpha^2+\beta^2) - 2\alpha^2 s_0(\alpha - 2\beta)\}(b^i y^j - b^j y^i) = 0 \quad \textbf{(3.3)}$$

Suppose that F^n is a Douglas space, that is, B^{ij} are *hp* (3). Separating rational and irrational terms of y^i in (3.3), then, we get the following two equations:

$$\{\alpha^2(1-2b^2)+3\beta^2\}\{(\alpha^2+\beta^2)B^{ij} + 2\alpha^2\beta(s_0^i y^j - s_0^j y^i)\} + \alpha^2\{r_{00}(\alpha^2+\beta^2) + 4\beta s_0\alpha^2\}(b^i y^j - b^j y^i) = 0 \quad \textbf{(3.4)}$$

and

$$2s_0\alpha^4(b^i y^j - b^j y^i) + \alpha^2\{\alpha^2(1-2b^2)+3\beta^2\}(s_0^i y^j - s_0^j y^i) = 0. \quad \textbf{(3.5)}$$

Substituting (3.5) in (3.4), we have

$$\{\alpha^2(1-2b^2)+3\beta^2\}(\alpha^2+\beta^2)B^{ij} + \alpha^2 r_{00}(\alpha^2+\beta^2)(b^i y^j - b^j y^i) = 0 \quad \textbf{(3.6)}$$

Only the term $3\beta^4 B^{ij}$ of (3.6) does not contain α^2.

Hence, we must have *hp* (5), v_5^{ij} satisfying

$$3\beta^4 B^{ij} = \alpha^2 v_5^{ij} \quad \textbf{(3.7)}$$

Now, we study the following two cases:

Case (i)

$\alpha^2 \not\equiv 0 (mod\ \beta)$

In this case, (3.7) is reduced to $B^{ij} = \alpha^2 v^{ij}$, where v^{ij} are *hp*(1). Thus, (3.6) gives

$$\{\alpha^2(1-2b^2)+3\beta^2\}v^{ij} + r_{00}(b^i y^j - b^j y^i) = 0 \quad \textbf{(3.8)}$$

Transvecting this by $b_i y_j$, where $y_j = a_{jk}y^k$, we have

$$\alpha^2\{(1-2b^2)v^{ij}b_i y_j + b^2 r_{00}\} = \beta^2(r_{00} - 3v^{ij}b_i y_j) \quad \textbf{(3.9)}$$

Since $\alpha^2 \not\equiv 0 (mod\ \beta)$, then there exists a function *h(x)*satisfying

$$(1-2b^2)v^{ij}b_i y_j + b^2 r_{00} = h(x)\beta^2,$$

$r_{00} - 3v^{ij}b_i y_j = h(x)\alpha^2.$

Eliminating $v^{ij}b_i y_j$ from the above two equations, we obtain

$$(1 + b^2)r_{00} = h(x)\{(1 - 2b^2)\alpha^2 + 3\beta^2\} \quad \textbf{(3.10)}$$

From (3.10), we get

$$b_{i|j} = k\{(-1 + 2b^2)a_{ij} - 3b_i b_j\} \quad \textbf{(3.11)}$$

Where $k = -h(x)/(1 + b^2)$. Hence b_i is a gradient vector.

Conversely, if (3.11) holds, then $s_{ij} = 0$ and we get (3.10). Therefore, (3.3) is written as follows:

$$B^{ij} = k\{\alpha^2(b^i y^j - b^j y^i)\}$$

Which is *hp* (3), that is, F^n is a Douglas space.

Case (ii)

$\alpha^2 \equiv 0(mod\, \beta).$

In this case, there exists 1-form δ such that $\alpha^2 = \delta\beta$, $b^2 = 0$ and by lemma 2.1, the dimension is two. Therefore (3.7) is reduced to $B^{ij} = \delta w_2^{ij}$, where w_2^{ij} are *hp*(2). Thus the equation (3.5) leads to

$$2s_0\delta(b^i y^j - b^j y^i) + (\delta + 3\beta)(s_0^i y^j - s_0^j y^i) = 0$$

Transvecting the above equation by $y_i b_j$, we have $s_0 = 0$. Substituting $s_0 = 0$ in the above equation, we have $s_{ij} = 0$. Therefore, (3.6) reduces to

$$(\delta + 3\beta)w_2^{ij} + r_{00}(b^i y^j - b^j y^i) = 0$$

Transvecting the above equation by $b_i y_j$, wec get

$$(\delta + 3\beta)w_2^{ij}b_i y_j - r_{00}\beta^2 = 0$$

Which is written as,

$$\delta w_2^{ij}b_i y_j = \beta(\beta r_{00} - 3w_2^{ij}b_i y_j)$$

Therefore, there exists an *hp* (2), $\lambda = \lambda_{ij}(x)y^i y^j$ such that

$$w_2^{ij}b_i y_j = \beta\lambda\,, \;\; \beta r_{00} - 3w_2^{ij}b_i y_j = \delta\lambda$$

Eliminating $w_2^{ij}b_i y_j$ from the above equations, we get

$$\beta r_{00} = \lambda(3\beta + \delta) \quad \textbf{(3.12)}$$

Which implies there exists an *hp* (1). $v_0 = v_i(x)y^i$ such that

$$r_{00} = v_0(3\beta + \delta),\;\; \lambda = v_0\beta \quad \textbf{(3.13)}$$

From r_{00} given by (3.13) and $s_{ij} = 0$, we get

$$b_{i|j} = \frac{1}{2}\{v_i(3b_j + d_j) + v_j(3b_i + d_i)\} \quad \textbf{(3.14)}$$

Hence b_i is a gradient vector.

Conversely, if (3.14) holds, then $s_{ij} = 0$, which implies $r_{00} = v_0(3\beta + \delta)$. Therefore, (3.3) is written as follows:

$B^{ij} = -v_0\delta(b^i y^j - b^j y^i)$

Which is *hp* (3). Therefore, F^n is a Douglas space. Thus, we have

Theorem 3.1

A Finsler space with a special (α, β)-metric $L = \alpha - \frac{\beta^2}{\alpha}$ is a Douglas space if and only if

- $\alpha^2 \not\equiv 0(\text{mod } \beta), b^2 \neq 1$: $b_{i|j}$ is written in the form (3.11).

$\alpha^2 \equiv 0(\text{mod } \beta)$: $n = 2$ and $b_{i|j}$ is written in the form (3.14), where $\alpha^2 = \beta\delta, \delta = d_i(x)y^i, v_o = v_i(x)y^i$.

REFERENCES

1. Benling Li, Yibing Shen and Zhongmin Shen, On a class of Douglas metrics, *Comm. Korean Math. Soc.,* 14(3) (1999), 535-544.
2. B.N Prasad. B.N Gupta and D.D. Singh, Conformal transformation in Finsler spaces with $(\alpha, \beta) -$ metric, *Indian J. Pure and Appl. Math.,* 18 (4) (1961), 290-301.
3. Hong-Suh Park and Eun-Seo Choi, Finsler spaces with an approximate Matsumoto metric of Douglas type, *Comm. Korean Math. Soc.,* 14 (3) (1999), 535-544.
4. M. Hashiguchi, S. Hojo and Matsumoto, On Landsberg spaces of dimension two with $(\alpha, \beta) -$ metric, *Tensor, N.S.,* 57 (1996), 145-153.
5. M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces,* Kaiseisha press, Otsu, Saikawa, Japan, (1986).
6. M. Matsumoto, The Berwald connection of a Finsler space with an $(\alpha, \beta) -$ metric, *Tensor, N.S.,* 50 (1991), 18-21.
7. M. Matsumoto, *On Finsler spaces with* $(\alpha, \beta) - metric$ *of Douglas type,* Tensor, N.S. 60(1998).
8. Pradeep Kumar, S.K. Narasimhamurthy, H.G. Nagaraja, S.T. Aveesh, On a special hypersurface of a Finsler space with $(\alpha, \beta) -$ metric, *Tbilisi Mathematical Journal,* 2 (2009), 51-60.
9. P.L. Antonelli, R.S. Ingarden and M. Matsumoto, The theory of sprays and Finsler spaces with applications in Physics and Biology, *Kluwer academic publishers,* Netherlands (1993).
10. S. Bacso, X. Cheng and Z. Shen, Curvature properties of $(\alpha, \beta) -$ metrics, *Advanced studies in Pure Mathematics, Math. Soc. Of Japan,* 48 (2007), 73-110.
11. S. Bacso and M. Matsumoto, On Finsler spaces of Douglas type. A generalization of the notion of Berawld spae, *Publ. Math. Debrecen,* 51 (1997), 385-406.

12. S. Bacso and I. Papp, On Landsberg spaces of dimension two with $(\alpha,\beta)-$metric, *Periodica Mathematica Hungarica,* 48 (1-2) (2004), 181-184.

13. S.K. Narasimhamurthy and D.M. Vasantha, Projective change between two Finsler spaces with $(\alpha,\beta)-$metric, *Kyunpook Mathematical Journal.*

14. S.K. Narasimhamurthy and G.N. Latha Kumari, On a hypersurface of a special Finsler space with a metric $L=\alpha+\frac{\beta^2}{\alpha}+\beta$, *ADJM,* 9 (1) (2010), 36-44.